Some Notes on Lattices

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August 3, 2004

1 Lattices

This document provides a gentle and very short introduction to Lattices. A lattice $(L, \wedge, \vee)$ consists of a set L and two operators $\wedge : L \times L \to L$, called *meet*, and $\vee : L \times L \to L$, called *join*, that satisfy the following for all $a, b, c \in L$:

$$
\begin{aligned}
a \vee a &= a & a \wedge a &= a & (1)\\
a \vee b &= b \vee a & a \wedge b &= b \wedge a & (2)\\
(a \vee b) \vee c &= a \vee (b \vee c) & (a \wedge b) \wedge c &= a \wedge (b \wedge c) & (3)\\
(a \vee b) \wedge c &= (a \wedge c) \vee (b \wedge c) & (a \wedge b) \vee c &= (a \vee c) \wedge (b \vee c) & (4)\\
(a \vee b) \wedge a &= a & (a \wedge b) \vee a &= a & (5)
\end{aligned}
$$

As a consequence of the definition above we can derive a number of properties. We can define a "partial order law" by

$$a \leq b \text{ iff } a \vee b = b \tag{6}$$

We can easily show that $a \leq b$ iff $a \wedge b = a$. Suppose $a \leq b$ then $b = a \vee b$ and

$$a \wedge b = a \wedge (a \vee b) = (a \vee a) \wedge (a \vee b) = a \vee (a \wedge b) = a \tag{7}$$

shows that $a \wedge b = a$, which shows one side of the *iff*. I will leave it to you to show that if $a \wedge b = a$ then we can show that $a \vee b = b$ and hence that $a \leq b$ (the other side of the *iff*.

It is fairly straight forward to show that if $a \leq b$ and $b \leq a$ then $a = b$.

The lattice always contains a unique *least* element (often called 0) which is the *meet* of all the elements of the lattice: $0 = \bigwedge_{a \in L} a$ In addition the lattice always contains a unique *maximal* element (often call 1) which is the *join* of all the elements of the lattice : $1 = \bigvee_{a \in L} a$. Note that for all $a \in L$ we have:

$$0 \leq a \leq 1 \tag{8}$$

Finally, the lattice contains the notion of *least upper bound* (*lub*) and *greatest lower bound* (*glb*). Suppose $a, b \in L$. Then there is a (unique) smallest element $u \in L$ such that $a \leq u$ and $b \leq u$. In fact we can show that $u = a \vee b$. Similarly, $l = a \wedge b$ is the largest element in the lattice such that $l \leq a$ and $l \leq b$.

Exersize: Show that the set of subsets of the set $A = \{1, 2, 3, 4, 5, 6\}$ forms a lattice under *set intersection* and *set union*. Determine the 0 and 1 of this lattice.

2 Some Examples

2.1 A Well-Ordered Lattice

Suppose $L = \{U, C, S, TS\}$ such that $U < C < S < TS$. If $a, b \in L$ we will define *meet* and *join* to be

$$a \wedge b = \min(a, b) \tag{9}$$
$$a \vee b = \max(a, b) \tag{10}$$

From this it is easy to see that $U \wedge TS = U$ and $U \vee TS = TS$. The definitions clearly satisfy equations $1 - 5$. It is clear why this is called a "well ordered" lattice.

2.2 A Lattice Based on Subsets

Suppose $L' = \{a, b, o, g\}$ is a set of 4 elements. Suppose $\mathcal{P}(L')$ is the *power set* (set of all subsets of) L'. Suppose we have for all elements $A, B \in \mathcal{P}(L')$, ($A, B \subseteq L'$)

$$A \wedge B = A \cap B \tag{11}$$
$$A \vee B = A \cup B \tag{12}$$

where $\cap$ is set intersection and $\cup$ is set union. Then it is easy to show that this forms a lattice.

2.3 Product Lattice

Suppose we construct the structure $(L \times L', \wedge, \vee)$ where for all $(l_1, l'_1), (l_2, l'_2 \in L \times L'$, we have:

$$(l_1, l'_1) \wedge (l_2, l'_2) = (l_1 \wedge l_2, l'_1 \cap l'_2) \tag{13}$$
$$(l_1, l'_1) \vee (l_2, l'_2) = (l_1 \vee l_2, l'_1 \cup l'_2) \tag{14}$$

where the operations are done "component wise."

It is easy to show that if the component operations form a lattice then the "product" will also form a lattice.

3 Showing equivalence of Lattice Definitions

In this section we show that there are several ways to define a lattice. We described one above, using the *join* and *meet* operators. Below we show an equivalent definition.

To do this, we first define a *poset* (partially ordered set). Suppose we have $(S, \leq)$ where S is some set and $\leq$ satisfies:

$$\begin{array}{ll} \forall s \in S : s \leq s & \text{(reflexive)} \\ \forall s, t \in S : s \leq t \text{ and } t \leq s \Rightarrow s = t & \text{(antisymmetry)} \\ \forall s, t, u \in S : s \leq t \text{ and } t \leq u \Rightarrow s \leq u & (\text{transitivity}) \end{array} \tag{15}$$

Some examples of partial orders include

1. The set of integers (positive and negative) with the natural integer comparison operator $\leq$.
2. The set of positive integers with the comparison operator defined $m \leq n \Leftrightarrow m$ divides n.[1]
3. The set of subsets (the power set) of a set S with $\subseteq$ as the comparison operator.

If, for every $s, t \in S$ we have either $s \leq t$ or $t \leq s$ then the set is *totally ordered*.

Suppose there exists an element $a \in S$ such that $\forall s \in S : a \leq s$. If such an element a exists, it is unique (i.e. there is only one of them). To show this, we suppose that there b also satisfies $\forall s \in S : b \leq s$. Then $a \leq b$ since a is a least element, and likewise since b is a least element

[1] *Divides* here means division without remainder. It is obvious that the requirements in equations 15 are satisfied. If we included both the positive and negative integers, then one of the conditions would not be satisfied. You should figure out which one.

$b \leq a$, but then by antisymmetry, $a = b$. We will denote such an element, if it exists, as the 0 of the lattice.

In the Example 1 above, there is no 0. In Example 2 the number "1" satisfies the 0 property, while in Example 3 the empty set ϕ satisfies the 0 property.

Using exactly dual arguements, we can deduce that if there is a largest element z such that for all $s \in S : s \leq z$, then z is unique. We call such an element "1", if it exists. Which of the examples above have a maximal element and which don't?[2]

Suppose $(S, \leq)$ is a partially ordered set and $X \subseteq S$. An "*upper bound*" of the set X is an element $a \in S$ such that $\forall x \in X : x \leq a$. For an arbitrary *poset* upper bounds may or may not exist. The lub (*least upper bound*) of the set X, written $lub(X)$, is an upper bound of X that is $\leq$ every other upper bound of X. Similarly we can define the glb (*greatest lower bound*) of a set X.

Using the same argument we used for showing that 0 is unique if it exists, we can show the both the $glb(X)$ and the $lub(X)$ are unique if they exist.

Suppose that $(S, \leq)$ is a partially ordered set such that every pair of elements of S has a glb and a lub. We will show that S is then a lattice. For all $s, t \in S$, we define

$$s \wedge t = glb(\{s, t\}) = glb(s, t) \tag{16}$$

$$s \vee t = lub(\{s, t\}) = lub(s, t) \tag{17}$$

To help up along, we need a couple of lemmas. First we need to show that for $s, t, u \in S$, $s \vee t = lub(s, t) \leq lub(s, t, u)$. We know that $s \leq lub(s, t, u)$ and $t \leq lub(s, t, u)$ so $lub(s, t, u)$ is an upper bound of both s and t. Hence, by definition of lub, we have $lub(s, t) \leq lub(s, t, u)$.

We also need to show that if $s \leq t$ then $lub(s, u) \leq lub(t, u)$. To show this, note that $s \leq t \leq lub(t, u)$ and $u \leq lub(t, u)$ so $lub(t, u)$ is an upper bound of $\{s, u\}$. Hence, by definition of lub, $lub(s, u) \leq lub(t, u)$.

We note that $lub(s, t) = lub(t, s)$. To show associativity of lub need to show that

$$lub(\{lub(\{s, t\}), u\}) = lub(\{s, lub(\{t, u\})\}).$$

One way to do this is to show that both sides are equal to $lub(\{s, t, u\})$. Since $lub(\{lub(\{s, t\}), u\})$ is an upper bound of $\{s, t, u\}$, we have $lub(\{s, t, u\}) \leq lub(\{lub(\{s, t\}), u\})$. Using our first lemma above, we have $lub(s, t) \leq lub(s, t, u)$ and $u \leq lub(s, t, u)$, so $lub(s, t, u)$ is an upper bound for $\{lub(s, t), u\}$, i.e we have $lub(lub(s, t), u) \leq lub(s, t, u)$.

This completes the proof of associativity.

[2]Note that having a minimal (0) element does not necessarily guarantee the existence of a maximal element.